Iteration of Holomorphic Maps

$f : \hat{\mathbb{C}} \longrightarrow \hat{\mathbb{C}}$ holomorphic

$$f(z) = \frac{p(z)}{q(z)} \quad , \; p, q \text{ polynomials coprime}$$

$$\deg f := \max\{\deg p, \deg q\}$$

$f^* : H^2(\hat{\mathbb{C}}, \mathbb{Z}) \circlearrowleft \qquad f^*(\omega) = d\,\omega$

$H^2(\hat{\mathbb{C}}, \mathbb{Z}) \cong \mathbb{Z}$

Pick $w \in \hat{\mathbb{C}}$

$$\{z : f(z) = w\} = \{z : p(z) - w\,q(z) = 0\}$$

Except for at most finitely many w,

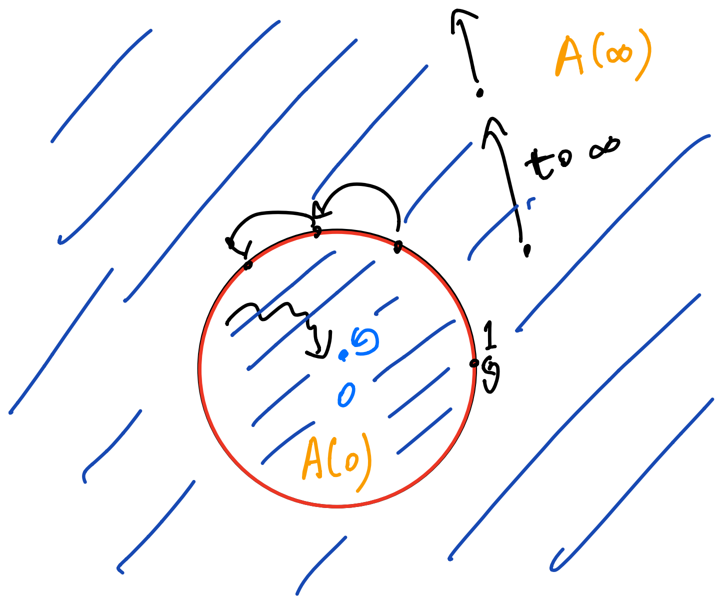
$$\#\{z : f(z) = w\} = \deg f$$

Note: a $f : \mathbb{C} \to \mathbb{C}$ holo need not be a rational function! $f(z) = e^z$.

E.g.:

$f(z) := z^2$

Pick $z \in \mathbb{C}$. What happens to $(f^n(z))_{n \geq 0}$?



Def.: A is the <u>basin of attraction</u> of z_0 if every $z \in A$ satisfies $f^n(z) \longrightarrow z_0$ as $n \to \infty$.

<u>Fatou set</u> "tame region"

<u>Julia set</u> "chaotic region"

In: $\{|z| < 1\}$: $f^n(z) \longrightarrow 0$ <u>Fatou set</u>

In $\{|z| > 1\}$: $f^n(z) \longrightarrow \infty$ <u>Fatou set</u>

In: $\{|z| = 1\}$ $f^n(z)\big|_n$ is not normal <u>Julia set</u>

Def.: A point $z \in \hat{\mathbb{C}}$ is in the FATOU SET for f if there exists an open $\mathcal{U} \ni z$ s.t. $\left(f^n\big|_{\mathcal{U}}\right)_{n \geq 0}$ is a normal family.

A point $z \in \hat{\mathbb{C}}$ is in the JULIA SET if it is not in the Fatou set.

E.g.: For $f(z) = z^2$

$$\Omega(f) = \{|z| < 1\} \cup \{|z| > 1\}$$

$$J(f) = \{|z| = 1\}$$

Rk 1 $\Omega(f)$ is open
$J(f)$ is closed

Def.: If $f : \mathbb{C} \to \mathbb{C}$ is a polynomial, we define the FILLED JULIA SET is

$$K(f) := \{ z \in \mathbb{C} : (f^n(z))_{n \geq 0} \text{ are bounded} \}$$

$$K(f) = \hat{\mathbb{C}} \setminus A(\infty)$$

Rk 2.: $J(f) = \partial K(f)$

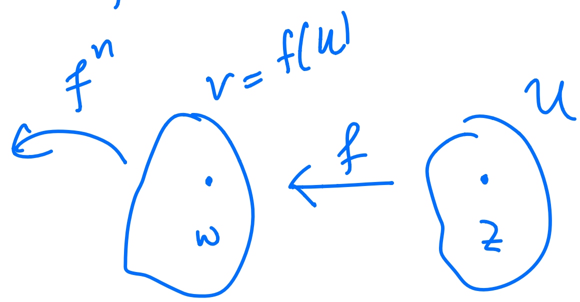
Fundamental Properties

L1 $f^{-1}(J(f)) = J(f)$ "invariance"

$f^{-1}(\Omega(f)) = \Omega(f)$

Pf If $z \in \Omega(f)$, $(f^n(z))_{n \geq 0}\big|_{\mathcal{U}}$ is normal.

$w = f(z)$



$$f^{n+1}\big|_{\mathcal{U}} = f \circ f^n\big|_{V}$$

then $f^{n+1}\big|_{V}$ is also normal $\Rightarrow w \in \Omega(f)$.

L2 $J(f^k) = J(f)$ for all $k \geq 1$.

$\Omega(f^k) = \Omega(f)$

Pf

① $z \in \Omega(f^k) \longleftrightarrow \exists\, \mathcal{U} \ni z : f^{nk}\big|_{\mathcal{U}}$ is normal

② $z \in \Omega(f) \longleftrightarrow \mathcal{U}' \ni z : f^n\big|_{\mathcal{U}}$ is normal

A subsequence of (f^{nk}) is also a subsequence of (f^n) so ① $\Rightarrow$ ②.

$$\{f^n\}_{n\geq 0} = \bigsqcup_{p=0}^{k-1} \{f^{mk+p}\}_{m\geq 0}$$

if a subsequence of $\{f^n\}$ converges, there is some p for which a subseq of $\{f^{mk+p}\}_{m\geq 0}$ also converges.

② $\Rightarrow$ ①.

Periodic Points

Def.: z is (purely) periodic of period p if $f^p(z) = z$.

The <u>MULTIPLIER</u> of f at z is

$$\lambda := (f^p)'(z)$$

We say z is

- ATTRACTING if $|\lambda| < 1$
- REPELLING if $|\lambda| > 1$
- INDIFFERENT if $|\lambda| = 1$
- PARABOLIC if λ is a root of 1.

L3 Attracting periodic points belong to the Fatou set. In fact, the entire basin of attraction of any attracting periodic point is contained in the Fatou set.

Pf.: $g = f^p$, p = period of z_0

z_0 attr. periodic pt.

in a nbd of z_0

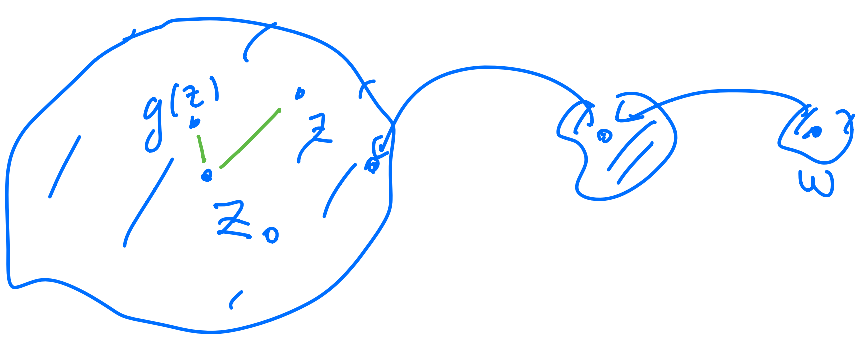
$$|g(z) - g(z_0)| \leq c\,|z - z_0|$$

for $c < 1$

$$|g(z) - z_0| \leq c\,|z - z_0|$$

By Banach fixed point theorem

$$g^n(z) \longrightarrow z_0 \quad \text{uniformly for } n \to \infty$$



L4 Repelling periodic pts lie in the Julia set.

Pf Suppose $z_0 = f(z_0)$ is repelling and $(f^n|_{\mathcal{U}}(z))_{n\geq 0}$ is normal.

Hence also $\left((f^n)'|_{\mathcal{U}} \right)_{n\geq 0}$ is normal

hence $\left((f^n)'(z_0) \right)_n$ has convergent subsequence, but

$$\left|(f^n)'(z_0)\right| = \left|\lambda^n\right| \longrightarrow \infty$$

which is a contradiction.

L5 Parabolic periodic points lie in the Julia set.

Pf Assume $f(0) = 0$ $\quad (p=1,\ z_0=0)$
$f'(0) = 1$

$$f(w) = w + a_q w^q + a_{q+1} w^{q+1} + \dots \quad (a_q \neq 0)$$

Hence

$$f^k(w) = w + k a_q w^q + O(w^{q+1})$$

$$f(f(w)) = f(w) + a_q (f(w))^q + O\left((f(w))^{q+1}\right)$$

$$= w + a_q w^q + O(w^{q+1}) +$$

$$+ a_q w^q + O(w^{q+1})$$

$$= w + 2a_q w^q + O(w^{q+1})$$

$$f'(w) = 1 + q a_q w^{q-1} + O(w^q)$$

$$f'(0) = 1$$

$$\boxed{f^k(w) = w + k a_q w^q + O(w^{q+1})}$$

$$(f^k)'(w) = 1 + k q a_q w^{q-1} + O(w^q)$$

$$(f^k)''(w) = k q (q-1) a_q w^{q-2} + O(w^{q-1})$$

$$\vdots$$

$$(f^k)^{(q)}(w) = k\, q!\, a_q + O(w)$$

$$\left|(f^k)^{(q)}(0)\right| = k\, q!\, |a_q| \to \infty \quad \text{as } k \to \infty.$$

Hence, (f^k) is not normal.

L6 The Julia set $J(f)$ is never empty if $\deg f \geq 2$.

Rk: The Fatou set may be empty! $J(f) = \hat{\mathbb{C}}$.

Pf.: If $J(f) = \emptyset$, then $(f^n)\big|_{\hat{\mathbb{C}}}$ is a normal family (normal is a local property)

Then $\exists\, n_j \to \infty$ s.t. $f^{n_j} \longrightarrow g$ on $\hat{\mathbb{C}}$. But then

$$\deg f^{n_j} = \deg g \qquad \text{for } j \gg 1.$$

Since degree is a homotopy invariant.

$$\deg f^{n_j} = (\deg f)^{n_j} \longrightarrow \infty$$

which is a contradiction.

Rk.: if $d = 1$, f is Möbius and then $\Omega(f)$ may equal $\hat{\mathbb{C}}$. (E.g. $f(z) = z$).

Def.: $GO(z,f)$ is the GRAND ORBIT of z under f

$$GO(z,f) := \{ w : \exists m,n \geq 0 : f^m(z) = f^n(w) \}$$

z is GRAND ORBIT FINITE or EXCEPTIONAL if $GO(z,f)$ is finite.

L7 The set of grand orbit finite points can have at most 2 elements. These points belong to the Fatou set.

E.g.: $f(z) = z^d$

$$\begin{tikzcd} 0 \arrow[loop above, "d"] & \infty \arrow[loop above, "d"] \end{tikzcd}$$

$f(z) = \dfrac{1}{z^d}$

$$\begin{tikzcd} 0 \arrow[r, bend left, "d"] & \infty \arrow[l, bend left, "d"] \end{tikzcd}$$